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Two-dimensional Lagrangian singularities and bifurcations of gradient lines I☆

G. Marelli

Department of Mathematics, Kyoto University, Kitashirakawa, Sakyo-ku, Kyoto 606-8502, Japan

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Abstract

Motivated by mirror symmetry, we consider a Lagrangian fibration $X \to B$ and Lagrangian maps $f : L \hookrightarrow X \to B$, when L has dimension 2, exhibiting an unstable singularity, and study how their caustic changes, in a neighbourhood of the unstable singularity, when slightly perturbed. The integral curves of ∇f_x, for $x \in B$, where $f_x(y) = f(y) - x \cdot y$, called "gradient lines", are then introduced, and a study of them, in order to analyze their bifurcation locus, is carried out.

MSC: 37G25; 53D12; 70K60

Keywords: Lagrangian singularities; Bifurcations theory; Mirror symmetry

1. Introduction

This is first of two papers motivated by the attempt of understanding some aspects of the homological mirror conjecture, when we assume the existence of dual torus fibrations. In it, we are concerned with the torus fibration $T^{2n} \to T^n$, as a first step in the direction towards generic Lagrangian torus fibrations. In this case mirror symmetry has been studied, under certain hypothesis, in papers such as [9,2,13,5,6], where the idea that mirror symmetry is a kind of Fourier–Mukai transform has been developed: given a Lagrangian submanifold L of X supporting a local system, under certain assumptions, a holomorphic bundle is obtained on a submanifold of $X^\vee$. In all these papers, a crucial hypothesis is that the caustic of L is empty, that is, the composition $L \hookrightarrow X \to B$

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E-mail address: marelli@kusm.kyoto-u.ac.jp.

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has no critical points. This paper, instead, takes the first steps in the direction of including the caustic. If $K \subset B$ denotes the caustic of L, we may think to restrict the fibration to $B \setminus K$: now L has no caustic and we may apply what is known in this case and obtain a holomorphic bundle on a certain submanifold of $X^\vee$ fibred over $B \setminus K$; however we realize that the holomorphic structure presents a monodromy which prevents from extending the holomorphic bundle over the points of the caustic K. As foreseen in [9], some quantum corrections must be performed in order to extend the holomorphic structure over points of the caustic. Quantum corrections or instanton effects are provided by pseudoholomorphic discs in X which bound L. Following [8], the fibre over $x \in B$ of the holomorphic bundle on $X^\vee$ is constructed as Lagrangian intersection Floer homology of L and of the Lagrangian fibre of X over x. This approach is equivalent to the Fourier–Mukai one when the caustic is empty, but, unlike this, has the advantage of naturally including pseudoholomorphic discs. Assuming that, near K, Lagrangian intersection Floer homology is equivalent to Morse homology defined through the generating function of L, assumption which still must be clarified and proved, enables us to study gradient lines of ∇f_x instead of pseudoholomorphic discs. This is the idea which leads the development of this paper. The theory of Lagrangian maps provides a classifications of Lagrangian singularities: in dimension 2 only folds and cusps are generic and stable; in dimension 3 other singularities appear, and so on. This suggests us to start by considering the case when L has dimension 2 and so X is the torus fibration $T^4 \to T^2$. If f is a (local) generating function of L, we plan to study, in a neighbourhood of a point $x \in K$, the gradient lines of the vector field ∇f_x, where $f_x(y) = f(y) - x \cdot y$, and their bifurcations, and with these to construct Morse homology. Troubles are given by those singularities which appear in dimension 2 as unstable, such as the elliptic umbilic. For these we study what happens to the caustic and, in the case of the elliptic umbilic, to the bifurcation locus of gradient lines, when a small perturbation is added to the generating function f. In fact, in the Fukaya category, Lagrangian maps are considered up to Hamiltonian equivalence, so we expect to recover the case of L having an unstable singularity by studying the case of a Lagrangian submanifold L' exhibiting a stable singularity and Hamiltonian equivalent to L. The analysis of possible phase portraits of ∇f_x should allow to construct the Morse complex, while the study of reciprocal positions of the caustic and bifurcation locus, providing morphisms of the Morse complex, should allow to construct a bundle whose holomorphic structure can be extended to the caustic.

In this paper, after reviewing in Section 2 some aspects about the classification of Lagrangian singularities, we study in Section 3 how unstable critical points of a Lagrangian map split when its generating function f is slightly perturbed. We first consider a map whose caustic is reduced to an elliptic umbilic, in a sense which we will specify, and see that a small perturbation modifies the caustic in a well-known curve known as tricuspoid. A similar analysis is sketched for maps exhibiting other unstable singularities, such as the hyperbolic umbilic, the swallow-tail and the parabolic umbilic, deducing some ideas about the way the problem could be faced. However, we recognize that, if we are interested in application to mirror symmetry, in dimension 2 the relevant singularities are the fold and the cusp, which are also stable, and the elliptic umbilic, the hyperbolic umbilic and the swallow-tail, which, as said, are stable and generic in dimension 3 though unstable in dimension 2. Singularities such as the parabolic umbilic, stable and generic in dimension at least 4, become relevant only when studying the problem in dimension 3. The analysis of the gradient lines of ∇f_x and their bifurcations are dicussed in Section 4 as a whole. This is essential to construct the Morse complex, in view of applications to mirror symmetry. In particular we study which kind of bifurcations can occur in a family of vector fields exhibiting only saddles and nodes, and, given a bifurcation diagram, when there exists a family of gradient vector fields providing that diagram. In [14], the sequel to this paper, we will analyze, in some specific cases, how the bifurcation locus changes when the generating function f is slightly perturbed.

2. Lagrangian submanifolds and their singularities

We recall some facts about Lagrangian submanifolds and their singularities, referring to [16] or [3] for details.

2.1. Lagrangian maps

Let (X, ω) be a symplectic $2n$-manifold, which will be denoted simply by X, and L an n-submanifold of X.

Definition 2.1. An immersion $g : L \to X$ is called Lagrangian immersion if $g^*\omega = 0$. If $L \subset X$ and the identical embedding is a Lagrangian immersion, then L is called Lagrangian submanifold.

Definition 2.2. A Lagrangian bundle is a symplectic manifold X endowed with a structure of smooth locally trivial bundle $\pi : X \to B$ over a base manifold B, all of whose fibres are Lagrangian submanifolds of X.

By Darboux's theorem, any point of X admits a neighbourhood with canonical coordinates, that is, coordinates $(y_1, x_1, \ldots, y_n, x_n)$ which are both canonical symplectic coordinates of X and such that the functions x_i are constant along the fibres of the bundles.

Definition 2.3. Let $\pi : X \to B$ be a Lagrangian bundle and $g : L \to X$ a Lagrangian immersion. We call Lagrangian map the composite map $\pi \circ g : L \to B$. The set K of critical values of $\pi \circ g$ is called caustic of the Lagrangian immersion g (or of the Lagrangian map $\pi \circ g$).

If non-empty, the caustic of a general Lagrangian map is an $(n-1)$-submanifold of B with singularities. A classification of singularities of Lagrangian maps is available and is obtained from the classification of singularities of smooth maps. Lagrangian maps are traced back to smooth functions by means of their generating function: let $L \hookrightarrow X$ be a Lagrangian submanifold, $p \in L$ a point, $\{y_i, x_i\}$ a system of canonical coordinates near p, then there is a set of indices $J \subset \{1, \ldots, n\}$ such that, if $I = \{1, \ldots, n\} \setminus J$, then $\{y_i, x_j\}$ are local coordinates of L near p, with $i \in I$ and $j \in J$, and there exists a smooth function f in the variables $\{y_i, x_j\}$, defined up to addition of a constant, such that L is determined by the equations

$$x_j = \frac{\partial f}{\partial y_i}, \qquad y_i = -\frac{\partial f}{\partial x_j} \tag{1}$$

conversely, given a function f as before, Eqs. (1) define a Lagrangian submanifold. The function f is called generating function of L.

Definition 2.4. Two Lagrangian bundles are said to be Lagrangian equivalent if there exists a bundle diffeomorphism between them, taking fibres to fibres, and mapping one symplectic form to the other. Analogously, two Lagrangian maps are said to be Lagrangian equivalent if there exists a Lagrangian equivalence of the corresponding fibre bundles sending the domain of the first map to that of the second.

If two maps are Lagrangian equivalent then their caustic are diffeomorphic. The converse of this statement is false.

In studying the classification of Lagrangian singularities, it is more convenient to enlarge the number of variables and describe a Lagrangian germ by a function of the enlarged set of variables and called generating family: for a given Lagrangian germ, a generating families is not uniquely determined, however the class defining equivalent Lagrangian germs can be described. If $f(y_i, x_j)$ is the generating function of a germ of a Lagrangian submanifold L, then

$$F(z, x) = f(z_i, x_j) + \langle z_j, x_j \rangle$$

is a generating family of L. Given F, then L can be described as the set

$$L = \{(y, x) | \exists z \text{ with } \partial F/\partial z = 0, y = \partial F/\partial x\}$$

and its caustic K as

$$K = \{x | \exists z \text{ with } \partial F/\partial z = 0, \det(\partial^2 F/\partial^2 x) = 0\}$$

Let D_0 be the group of germs at 0 of diffeomorphisms of $\mathbb{R}^n$ preserving 0.

Definition 2.5. Two germs f_1 and f_2 of functions at 0 are D_0-equivalent if there exists a germ $\phi \in D_0$ such that $f_1 = f_2 \circ \phi$.

Two germs f_1 and f_2 of functions at 0 are stably D_0-equivalent if there exists a germ $\phi \in D_0$ and a non-degenerate quadratic form Q in additional variables such that $f_1 = f_2 \circ \phi + Q$.

Theorem 2.6. *Germs of Lagrangian maps are Lagrangian equivalent if and only if their generating families are stably equivalent.*

2.2. The Whitney topology

Let X be a $2n$-symplectic manifold and $X \to B$ a Lagrangian bundle. Generating functions of Lagrangian maps are elements of $C^\infty(\mathbb{R}^n)$. We endow the space of smooth function $C^\infty(\mathbb{R}^n)$ with the Whitney C^∞ topology (see also [10,7]).

Definition 2.7. For every non-negative integer k, and for every subset $U \subset J^k(\mathbb{R}^n)$, where $J^k(\mathbb{R}^n)$ denotes the space of k-jets of smooth functions, let $M(U) = \{f \in C^\infty(\mathbb{R}^n) | j^k f(\mathbb{R}^n) \subset U\}$. The family of sets $\{M(U)\}$ forms a basis for the Whitney C^k topology on $C^\infty(\mathbb{R}^n)$. The Whitney C^∞ topology is the topology with basis $W = \cup_{k=0}^\infty W_k$, where W_k is the set of open subsets of $C^\infty(\mathbb{R}^n)$ in the Whitney C^k topology.

Endowed with the Whitney C^∞ topology, $C^\infty(\mathbb{R}^n)$ is a Baire space, so every residual subset is dense.

Definition 2.8. A Lagrangian map is said to be Lagrangian stable if every nearby Lagrangian map, in the Whitney topology, is Lagrangian equivalent to it.

It can be proved that a germ of a Lagrangian map given by a generating family F is Lagrangian stable if and only if F is a versal deformation of $f(y_i, 0)$, and that its caustic is a component of the bifurcation set of its generating family.

Definition 2.9. A property P of smooth functions in $C^\infty(\mathbb{R}^n)$ is generic if:

1. $C_P = \{f \in C^\infty(\mathbb{R}^n) | f \text{ satisfies } P\}$ contains a residual subset of $C^\infty(\mathbb{R}^n)$;

2. let $f \in C_P$ and suppose g is Lagrangian equivalent to f, then $g \in C_P$.

A quasi-norm, and so a metric, generating the Whitney C^∞ topology, can be defined on $C^\infty(\mathbb{R}^n)$ (see again [10,7] for details), so that it makes sense to talk of small perturbations of a function $f \in C^\infty(\mathbb{R}^n)$.

2.3. Classifications of Lagrangian singularities

According to Theorem 2.6, the problem of classifying Lagrangian singularities is reduced to classify singularities of functions up to stably D_0-equivalence. The next theorem explains what happens in low dimensions. For a list of normal forms see [3] or [16].

Theorem 2.10. *The germs of generic Lagrangian maps $L \hookrightarrow X \to B$, with L of dimension $n \leq 5$, are stable and belong to a finite number of classes of Lagrangian equivalence. When $n > 5$ moduli appear, which in higher dimensions become functional moduli. A classification of generic Lagrangian singularities exists for $n \leq 10$.*

When $n \leq 3$, the possible generating functions, denoted by letters A or D, together with an index which represents the Milnor number, are:

- $n \geq 1$

$$\text{the fold } A_2: \quad f(y_1) = y_1^3 \tag{2}$$

- $n \geq 2$

$$\text{the cusp } A_3: \quad f(y_1, x_2) = \pm y_1^4 + x_2 y_1^2 \tag{3}$$

- $n \geq 3$

$$\text{the swallow-tail } A_4: \quad f(y_1, x_2, x_3) = y_1^5 + x_2 y_1^3 + x_3 y_1^2 \tag{4}$$

$$\text{the hyperbolic umbilic or purse } D_4^+: \quad f(y_1, y_2, x_3) = y_1^3 + y_1 y_2^2 + x_3 y_1^2 \tag{5}$$

$$\text{the elliptic umbilic or pyramid } D_4^-: \quad f(y_1, y_2, x_3) = y_1^3 - y_1 y_2^2 + x_3 y_1^2 \tag{6}$$

3. Perturbations of two-dimensional unstable singularities

Let X be a 4-symplectic manifold and $X \to B$ a Lagrangian bundle. When Lagrangian submanifolds have dimension 2, only folds and cusps can appear locally as singularities of generic stable Lagrangian maps, however other singularities can appear as non-generic ones. In this case, such singularities are not stable and break in folds and cusps as a consequence of any generic perturbations.

Suppose that a Lagrangian map has an unstable critical point at p and we want to study how this singularity decomposes after a small perturbation. To this purpose, consider a small perturbation f', which we can suppose supported on a disc D containing p. This defines a new generating function $\tilde{f} = f + f'$ and a Lagrangian submanifold $\tilde{L}$; L and $\tilde{L}$ coincide outside a compact subset D' of X and their caustics differ only in $f(D) \subset B$.

Being interested, as a first step, in a Lagrangian torus fibration with two-dimensional smooth fibres, and since the decomposition of an unstable singularity is a local problem, we can consider the Lagrangian fibration $\mathbb{R}^4 \to \mathbb{R}^2$. We use coordinates (x_1, x_2) on the base and (y_1, y_2) on the fibres.

3.1. The elliptic umbilic

We refer to the generating function

$$f(y_1, y_2) = \tfrac{1}{3} y_1^3 - 2 y_1 y_2^2 \tag{7}$$

defining the Lagrangian map

$$x_1 = y_1^2 - y_2^2, \qquad x_2 = -2 y_1 y_2 \tag{8}$$

as the elliptic umbilic in dimension 2. It has an unique critical point, the origin $(0, 0)$ of the (y_1, y_2)-plane: it is neither a fold nor a cusp, so it is unstable. The caustic is the subset $\{(0, 0)\}$ of the (x_1, x_2)-plane. To study how it splits when f is slightly perturbed, we add a perturbation f' and consider the new generating function $\tilde{f} = f + f'$.

Proposition 3.1. *For a generic and small* f', $\tilde{f}$ *has caustic diffeomorphic to a tricuspoid, the curve shown in* Fig. 1 (*see* [3] *for a definition of tricuspoid*).

Having only folds and cusps, the tricuspoid is stable. If $f'(y_1, y_2) = \frac{\epsilon}{2}(a y_1^2 + b y_1 y_2 + c y_2^2)$ is a generic polynomial of degree 2, the critical locus turns out to be a circle in the (y_1, y_2)-plane with centre

$$C = \left(-\frac{\epsilon}{4}(a - c), \frac{\epsilon}{4} b\right)$$

and radius

$$\frac{\epsilon}{4}|a + c|$$

in this case the caustic is a tricuspoid and can be explicitly computed (see [3]).

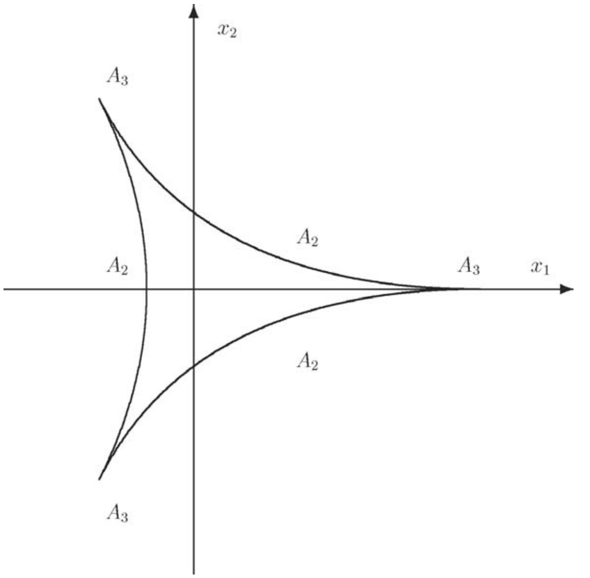

Fig. 1. The tricuspoid.

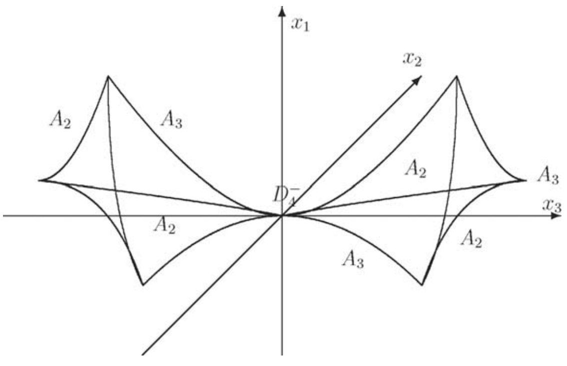

Fig. 2. The pyramid.

Proof. By hypothesis, in the Whitney topology of $C^\infty(\mathbb{R}^2)$ $\tilde{f}$ lies in a small neighbourhood of f, so, if T is a tubular neighbourhood of graph$(f) \subset \mathbb{R}^2 \times \mathbb{R}$, we can identify $\tilde{f}$ with a section of $C^\infty(T)$ and find a deformation $\tilde{f}$ from f to $\tilde{f}$. The Milnor number of f is 4, thus a versal deformation F of f has four parameters and can be written as $F(y_1, y_2) = f(y_1, y_2) + a_0 + a_1 y_1 + a_2 y_2 + a_3 y_1^2$. By definition, any other deformation G of f is obtained from F as $G(y, \lambda) = F(H(y, \lambda), \Phi(\lambda))$, where $y = (y_1, y_2)$, λ represents the parameters of the deformation, H is a family of diffeomorphisms parameterized by λ and Φ is a smooth function of λ. Observe that F is a generating family of the elliptic umbilic in dimension 3 (in fact f is the normal form of the singularities D_4^-): it defines a generating function (see Eq. (6))

$$\bar{f}(y_1, y_2, x_3) = y_1^3 - y_1 y_2^2 + x_3 y_1^2$$

and a Lagrangian map

$$x_1 = y_1^2 - y_2^2 + 2x_3 y_1, \qquad x_2 = -2y_1 y_2, \qquad y_3 = -y_1^2$$

whose caustic K_F, showed in Fig. 2, is the well-known pyramid.

Note that f is recovered from $\bar{f}$ by setting $x_3 = 0$, so that the caustic K_f of f can be identified with the intersection $K_F \cap \{x_3 = 0\}$ between the pyramid and the plane $x_3 = 0$. Observe instead that the intersection $K_F \cap \{x_3 = t\}$ is, for $t \neq 0$, a tricuspoid. For f' sufficiently small, $\tilde{f}$ is a small deformation of the elliptic umbilic in dimension 3, and being this stable, it follows that the caustic $K_{\tilde{f}}$ of $\tilde{f}$, in suitable coordinates x_1', x_2' and x_3', is still the pyramid. On the other hand, the versality of F ensures the existence of a map Φ such that $\Phi(x_3) = x_3'$ and relating, as explained, $\tilde{f}$ to F. Choosing f' sufficiently small, Φ will be enough close to the identity, in the Whitney topology, to be injective. Since $K_{\tilde{f}} = K_{\tilde{f} \cap \{x_3 = 0\}}$, it follows that the caustic $K_{\tilde{f}}$ of $\tilde{f}$ is generically diffeomorphic to a tricuspoid. $\square$

3.2. The hyperbolic umbilic

We refer to the generating function

$$f(y_1, y_2) = \tfrac{1}{3}(y_1^3 + y_2^3) \tag{9}$$

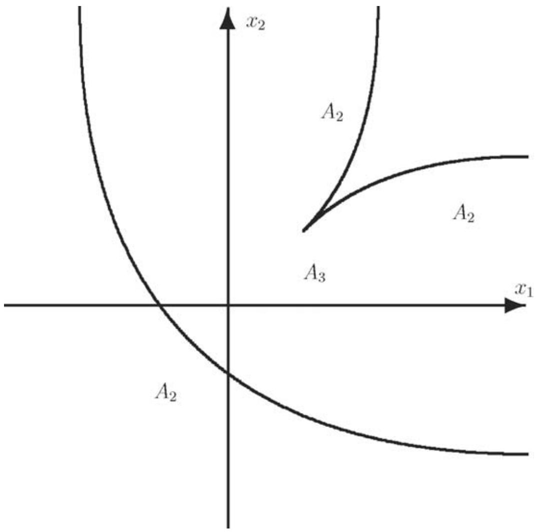

Fig. 3. The caustic of a small perturbation of the hyperbolic umbilic.

whose associated Lagrangian map is

$$x_1 = y_1^2, \qquad x_2 = y_2^2$$

as the hyperbolic umbilic in dimension 2. The critical locus is given by $y_1 y_2 = 0$. The caustic is the set $\{x_1 x_2 = 0 : x_1, x_2 \geq 0\}$.

Proposition 3.2. *A generic small perturbation of the hyperbolic umbilic in dimension 2 has a caustic diffeomorphic to the non-connected subset shown in* Fig. 3.

Proof. The argument is the same as the one used in the proof of Proposition 3.1. □

3.3. Other singularities

In dimension 2 we can consider other unstable germs of functions and try to study how their caustics change, when slightly perturbed, by using their versal deformations. As seen in the previous subsections, being the elliptic and hyperbolic umbilics, and also the swallow-tail (see [3]), stable in dimension 3, the study of the generating functions (7) and (9), in dimension 2, was recovered from the analysis of the generating functions (6) and (5), in dimension 3, by fixing one parameter. Instead, consider, for instance, the parabolic umbilic (see [4]), which is stable in dimension 4: it is necessary to fix two parameters to recover the case of dimension 2 from the stable case in dimension 4. So, if interested in some applications to mirror symmetry when the total space of the fibration has complex dimension 2, it seems to be not so relevant to consider those unstable singularities, such as the parabolic umbilic, whose versal deformations define stable singularities in dimension greater than 3: indeed, a generic orbit of Hamiltonian equivalence containing a Lagrangian map exhibiting an unstable singularity, such as the elliptic or hyperbolic umbilic, after a small perturbation, still will contain a Lagrangian map with such singularity; this is no longer true if the singularity is, for example, a parabolic umbilic.

4. Gradient lines and their bifurcations

4.1. Gradient lines of a Lagrangian map

Given a Lagrangian map $L \hookrightarrow X \to B$ with generating function *f*, where we always assume $X = \mathbb{R}^4$ and $B = \mathbb{R}^2$, and fixed a metric on *X*, we define a family of functions $f_x : \mathbb{R}^2 \to \mathbb{R}$, parameterized by $x \in B$, as $f_x(y) = f(y) - x \cdot y$, and consider a dynamical system on each fibre $X_x = \mathbb{R}^2$ of *X*, over *x*, as follows:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \nabla f_x \tag{10}$$

where ∇ is the gradient induced by the metric on *X*.

Definition 4.1. A curve $y:(a, b) \to X_x$, with $a, b \in \mathbb{R} \cup \{+\infty, -\infty\}$, is called a gradient line if it is a solution of (10).

Note that the set of critical points of ∇f_x coincides with the intersection $L \cap X_x$.

Lemma 4.2. *If $x \notin K$, where K is the caustic of L, then f_x has only non-degenerate critical points (in other words, f_x is a Morse function).*

Proof. If *y* is a critical point of f_x, then $\nabla f_x(y) = \nabla f(y) - x = 0$. If $x \notin K$ then $Hf(y) = Hf_x(y)$ has maximal rank. □

Gradient vector fields share the following feature.

Lemma 4.3. *If f has only finitely many non-degenerate critical points, then ∇f has finitely many fixed points all of which are hyperbolic and no other periodic orbits.*

Proof. See [15]. □

The Morse index of a non-degenerate critical point *y* of *f* is defined as the number of negative eigenvalues of the Hessian $Hf(y)$. In dimension 2, Lemma 4.3 implies that if *x* does not belong to the caustic, we expect as critical points of ∇f_x only unstable nodes, saddles and stable nodes, identified by Morse index respectively equal to 0, 1 and 2.

4.2. Bifurcation points of a Lagrangian map

Definition 4.4. A point $x \in B$ is a bifurcation point of *f* if and only if $x \notin K$ and ∇f_x is not Morse–Smale (see [7] or [12] or [15] for the definition of Morse–Smale vector field).

Corollary 4.5. *Let x be a bifurcation point, then there exist two critical points y_1 and y_2 of ∇f_x such that $W^{\mathrm{u}}(y_1)$ and $W^{\mathrm{s}}(y_2)$ do not intersect transversely, where W^{u} and W^{s} denote respectively the unstable and stable manifold of critical points.*

Proof. It is a direct consequence of the definition of Morse–Smale vector field and of the fact that $x \notin K$ and that ∇f_x is a gradient vector field. □

Remark 4.6. Observe that for vector fields on 2-manifolds, the Morse–Smale condition is equivalent to structural stability.

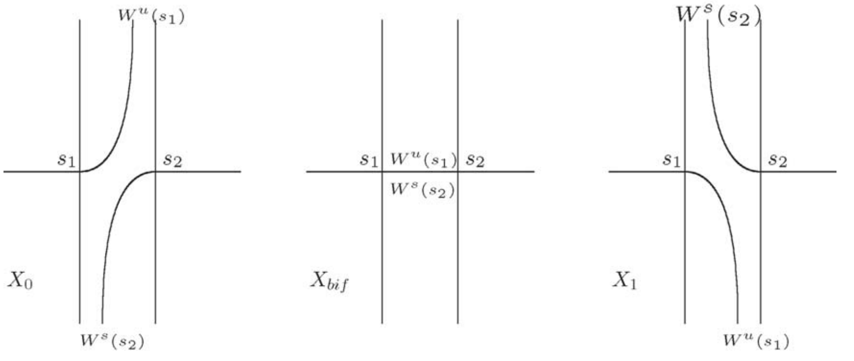

Fig. 4. A saddle-to-saddle separatrix.

In dimension 2, the critical points in Corollary 4.5 are saddles. Since the stable and unstable manifolds of a saddle are each the union of two of the four separatrices of the saddle, a non-transversal intersection of $W^{\mathrm{u}}(y_1)$ and $W^{\mathrm{s}}(y_2)$ means that y_1 and y_2 have a common separatrix, or, in other words, that there is a gradient line from y_1 to y_2. We call saddle-to-saddle separatrix such homoclinic orbit.

Proposition 4.7. *A saddle-to-saddle separatrix is not structurally stable.*

Proof. See [1]. □

Fig. 4 shows the bifurcation given by a saddle-to-saddle separatrix from s_1 to s_2.

Observe that the structurally stable vector fields X_0 and X_1, though orbitally equivalent, are not orbitally equivalent under deformations, in the sense that, if ϕ is the homeomorphism of the plane mapping the phase portrait of X_0 to the phase portrait of X_1 and respecting the sense of the flow, and if Φ a homotopy between the identity and ϕ, with parameter space [0, 1], then there exists $t \in (0, 1)$ such that $\Phi(, t)$ is not a homeomorphism (in other words, as it is qualitatively evident, a continuous deformation of the phase portrait of X_0 to the one of X_1, respecting the direction of the flow, contains the phase portrait of the unstable vector field X_{bif}). Thus, for a generic family of vector fields exhibiting two saddles, near an element having a saddle-to-saddle separatrix, there are two classes of vector fields up to orbitally equivalence under deformations.

Denote by $\mathcal{M}(y_1, y_2)$ the moduli space of unparameterized gradient lines from a critical point y_1 to a critical point y_2.

Proposition 4.8. *If ∇f_x is Morse-Smale and $\mathcal{M}(y_1, y_2) \neq \emptyset$ then*

$$\dim \mathcal{M}(y_1, y_2) = \mathrm{ind}(y_1) - \mathrm{ind}(y_2) - 1$$

Proof. See for example [7]. □

This implies that gradient lines from y_1 to y_2 exist generically only if the Morse index of y_1 is greater than the Morse index of y_2, and they are stable.

Observe that, also for $x \in K$, the vector field ∇f_x is not Morse–Smale: what happens is that the nature or the number of critical points of ∇f_x change. These bifurcations are called local bifurcations, because it is enough to study the vector field in a neighbourhood of the degenerate bifurcating critical points. Instead, those bifurcations involving a lack of transversality between the stable and unstable manifolds of two critical points, as in the case of a saddle-to-saddle separatrix, are called global, since involving global properties of the flow of the field ∇f_x (see [11]).

Definition 4.9. The bifurcation locus $\mathcal{B}$ of f is the set of bifurcation points of f. The diagram containing the caustic K and the bifurcation locus $\mathcal{B}$ of f in $B = \mathbb{R}^2$ is the bifurcation diagram of f.

Each point x of a bifurcation diagram gives information about critical points and existence of saddle-to-saddle separatrices of ∇f_x. Far from the caustic K, the vector field ∇f_x exhibits a certain number of saddles $s_1(x), \ldots, s_n(x)$, so we can define components $\mathcal{B}_{i,j}$ of the bifurcation locus $\mathcal{B}$ as the set of points x such that ∇f_x exhibits a gradient line $\gamma_{s_i(x)s_j(x)}$ from $s_i(x)$ to $s_j(x)$.

Proposition 4.10. *Far from K and from other components of $\mathcal{B}$, $\mathcal{B}_{i,j}$, if non-empty, is an immersed submanifold of codimension 1.*

Proof. Let $S(s_i(x_0))$ and $S(s_j(x_0))$ be respectively the separatrices of $s_i(x_0)$ and of $s_j(x_0)$ which intersect, at $x_0 \in \mathcal{B}_{i,j}$, in the gradient line $\gamma_{s_i(x_0)s_j(x_0)}$. Consider a neighbourhood $N(x_0)$ of x_0 such that $N(x_0)$ does not intersect K or other components of $\mathcal{B}$ different from $\mathcal{B}_{i,j}$, then, for all $x \in N(x_0)$, the vector field ∇f_x, if structurally stable, belongs to two distinct classes $\mathcal{V}_1$ and $\mathcal{V}_2$ up to orbital equivalence under deformations. Define $\psi : N(x_0) \to \mathbb{R}$ as

$$\psi(x) = \begin{cases} \mathrm{dist}(S(s_i(x)), S(s_j(x)))^2 & x \in \mathcal{V}_1 \\ -\mathrm{dist}(S(s_i(x)), S(s_j(x)))^2 & x \in \mathcal{V}_2 \end{cases}$$

Note that ψ is smooth everywhere, because the family f_x depends smoothly on x, and that, if non-empty, $\mathcal{B}_{i,j} = \psi^{-1}(0)$ (this is true because $N(x_0)$ does not intersect K or other components of $\mathcal{B}$ different from $\mathcal{B}_{i,j}$: in fact if a saddle $s_k(x)$, with $k \neq i, j$, were a limit point of both $S(s_i(x))$ and $S(s_j(x))$, then $\psi(x) = 0$ though there is no gradient line from $s_i(x)$ to $s_j(x)$). Generically, ψ is a Morse function, thus $\mathcal{B}$ is an immersed submanifold of $N(x_0)$; moreover, far from its critical points, ψ is transversal to $0 \in \mathbb{R}$, so $\mathcal{B}_{i,j}$ is a submanifold of $N(x_0)$ of codimension 1. □

In a similar way we can define subsets $\mathcal{B}_{(i,j),(k,l)}$ of $\mathcal{B}$ as the set of points $x \in B$ where ∇f_x exhibits both the exceptional gradient lines $\gamma_{s_i(x)s_j(x)}$ and $\gamma_{s_k(x)s_l(x)}$.

Corollary 4.11. *Far from K and from other components of $\mathcal{B} \setminus (\mathcal{B}_{i,j} \cup \mathcal{B}_{k,l})$, $\mathcal{B}_{(i,j),(k,l)}$, if non-empty, is an immersed submanifold of codimension 2.*

Proof. Define, in a neighbourhood $N(x_0)$ of a point $x \in \mathcal{B}_{(i,j),(k,l)}$ which is far from K and other components of $\mathcal{B} \setminus (\mathcal{B}_{i,j} \cup \mathcal{B}_{k,l})$, a function $\psi : B \to \mathbb{R}^2$ as $\psi(x) = (\psi_{ij}, \psi_{kl})$, where ψ_{ij} and ψ_{kl} are as in the proof of Proposition 4.10, and note that $\mathcal{B}_{(i,j),(k,l)} = \psi^{-1}(0)$ and $\{0\}$ has codimension 2 in $\mathbb{R}^2$. □

For a generic f, $\mathcal{B}_{(i,j),(k,l)} = \mathcal{B}_{i,j} \cap \mathcal{B}_{k,l}$. It is clear that three exceptional gradient lines in the same phase portrait is a bifurcation of codimension greater than 2, so, generically, it does not occur in dimension 2. Therefore, $\mathcal{B}$ can be decomposed into strata $\mathcal{B}_{i,j}$ and $\mathcal{B}_{(i,j),(k,l)}$, whose codimension is respectively 1 and 2.

Whether exceptional gradient lines appear or not, or in other words, whether the subsets $\mathcal{B}_{i,j}$ and $\mathcal{B}_{(i,j),(k,l)}$ are non-empty, it depends on the family of vector fields. We will analyze those cases which we need to study the bifurcation diagram of the cusp and of the elliptic umbilic. Since we are dealing with the family ∇f_x, we may assume that all the elements of the family are gradient vector fields, in order to avoid troubles with periodic orbits.

4.3. A family of vector fields with two saddles

Consider a 2-parameters family of gradient vector fields X_x exhibiting two saddles s_1 and s_2. Consider a structurally stable element X_s of the family. By definition of stability, there exists a neighbourhood U of s such that, for every $t \in U$, X_t is conjugated to X_s. On the boundary ∂U of U we can expect to meet a bifurcation point b, where X_b presents a saddle-to-saddle separatrix.

Proposition 4.12. *A point $b \in \mathcal{B} \cap \partial U$ can belong to either $\mathcal{B}_{1,2}$ or $\mathcal{B}_{2,1}$.*

Proof. The saddle-to saddle separatrix of X_b can be given by either $W^{\mathrm{u}}(s_1) \cap W^{\mathrm{s}}(s_2)$ or by $W^{\mathrm{s}}(s_1) \cap W^{\mathrm{u}}(s_2)$, which means that b belongs respectively to $\mathcal{B}_{1,2}$ or to $\mathcal{B}_{2,1}$. □

By Proposition 4.10, both $\mathcal{B}_{1,2}$ and $\mathcal{B}_{2,1}$ have codimension 1.

Lemma 4.13. *$\mathcal{B}_{1,2} \cap \mathcal{B}_{2,1} = \emptyset$. The intersection of two components $\mathcal{B}^1_{ij}$ and $\mathcal{B}^2_{ij}$ of $\mathcal{B}_{ij}$ is non-empty provided the saddle-to-saddle separatrices γ^1_{ij} and γ^2_{ij}, appearing respectively at points of $\mathcal{B}^1_{ij}$ and $\mathcal{B}^2_{ij}$, are obtained as intersection of the same pair of separatrices of s_1 and s_2.*

Proof. Let $b \in \mathcal{B}_{1,2} \cap \mathcal{B}_{2,1}$, then X_b exhibits two saddles and, between them, two saddle-to-saddle separatrices with opposite directions; consider in $\mathbb{R}^2$ a close curve C containing s_1 and s_2, then the Poincaré index $\mathrm{ind}_{\mathcal{P}}(C)$ of C would be equal to -1, while, on the other hand, $\mathrm{ind}_{\mathcal{P}}(C) = \mathrm{ind}_{\mathcal{P}}(s_1) + \mathrm{ind}_{\mathcal{P}}(s_2) = -2$. This proves the first statement.

For $t \in \mathcal{B}^1_{ij} \cap \mathcal{B}^2_{ij}$, suppose the gradient lines γ^1_{ij} and γ^2_{ij} of X_t are obtained as intersection of different pairs of separatrices of the saddles, then the phase portrait of X_t exhibits two exceptional gradient lines between the two saddles, giving a contradiction as shown in the first part of the proof. Otherwise, no contradiction arises at t, since only one saddle-to-saddle separatrix appears in the phase portrait of X_t. Moreover, if α, β, γ and δ denote the four subsets determined in $\mathbb{R}^2$ by $\mathcal{B}^1_{ij}$ and $\mathcal{B}^2_{ij}$, then the two classes of orbitally equivalent vector fields under deformation are given by $x \in \alpha \cup \gamma$ and $x \in \beta \cup \delta$ (see Fig. 5). □

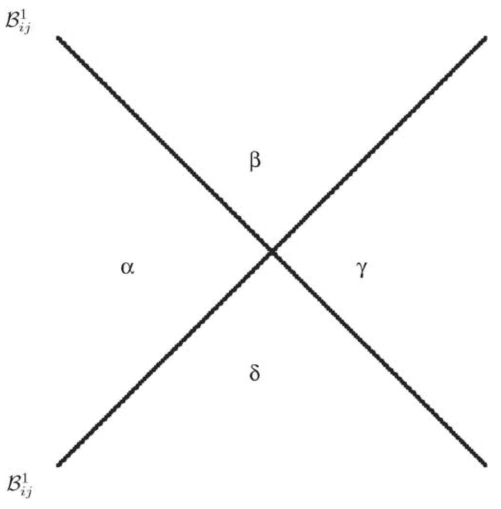

Fig. 5. The intersection of two components of $\mathcal{B}_{ij}$.

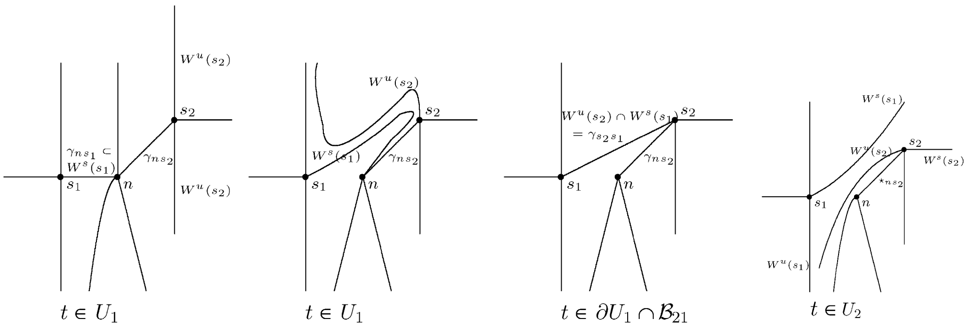

Fig. 6. The bifurcation from U_1 to U_2.

4.4. A family of vector fields with two saddles and one node

Consider a 2-parameters family of gradient vector fields X_x with two saddles s_1 and s_2 and an unstable node n. We want to understand which kind of bifurcations, that is, which kind of saddle-to-saddle separatrices, the family can exhibit. Consider, if existing, a structurally stable element X_s of the family, such that its phase portrait contains both the gradient lines γ_{ns_1} and γ_{ns_2}, from n to respectively s_1 and s_2. Structural stability ensures the existence of an open connected neighbourhood U_1 of s in $\mathbb{R}^2$ such that, for every $t \in U_1$, the phase portrait of X_t is orbitally equivalent to the phase portrait of X_s. Among such neighbourhoods of s we can assume U_1 to be maximal.

Proposition 4.14. *If $\mathcal{B} \cap \partial U_1 \neq \emptyset$, a point $t \in \mathcal{B} \cap \partial U_1$ belongs to either $\mathcal{B}_{1,2}$ or $\mathcal{B}_{2,1}$: the saddle-to-saddle separatrix $\gamma_{s_i s_j}$ is obtained as intersection of γ_{ns_j} with one of the two components of $W^{\mathrm{u}}(s_i)$; at t, where $\gamma_{s_i s_j}$ appears, γ_{ns_j} breaks.*

Proof. For $t \in \mathcal{B} \cap \partial U_1$, the saddle-to-saddle separatrices which can be exhibited in the phase portrait of X_t are $\gamma_{s_1 s_2}$ and $\gamma_{s_2 s_1}$, implying that t belongs respectively to $\mathcal{B}_{1,2}$ and $\mathcal{B}_{2,1}$. Consider, for example, $\gamma_{s_2 s_1}$ ($\gamma_{s_1 s_2}$ can be treated similarly): $\gamma_{s_2 s_1} = W^{\mathrm{u}}(s_2) \cap W^{\mathrm{s}}(s_1)$; as shown in Fig. 6 the only connected component of $W^{\mathrm{s}}(s_1)$ which can intersect $W^{\mathrm{u}}(s_2)$ is γ_{ns_1}, while both the unstable separatrices of s_2, the connected component of $W^{\mathrm{u}}(s_2)$, can intersect $W^{\mathrm{s}}(s_1)$. This implies that when, at the bifurcation point t, $\gamma_{s_2 s_1}$ appears, γ_{ns_1} breaks. □

Let U_2 be a (maximal) open connected subset such that X_t is structurally stable for all $t \in U_2$ and $\mathcal{B}_{ij} \cap \partial U_1 \cap \partial U_2 \neq \emptyset$.

Proposition 4.15. *For all $t \in U_2$, the phase portrait of X_t does not exhibit the gradient line γ_{ns_j}. At a point $t \in \partial U_2 \cap \mathcal{B}$, two pairs of separatrices, one of s_i and one of s_j, can intersect in a saddle-to-saddle separatrix: in one case, described in Proposition* 4.14 *and shown in* Fig. 6, *t belongs to $\mathcal{B}_{ij}$, moreover, after the bifurcation, the line γ_{ns_j} appears in the phase portrait of X_t; in the other case, analyzed in Section* 4.3 *for a family of vector fields with two saddles, t belongs to $\mathcal{B}_{ji}$, moreover after the bifurcation no gradient line γ_{ns_j} appears in the phase portrait of X_t* (*see* Fig. 7).

Proof. That for $t \in U_2$ the phase portrait of X_t does not contain γ_{ns_j} is a consequence of Proposition 4.14. Setting for simplicity $i = 2$ and $j = 1$ as in the proof of Proposition 4.14, the two pair of separatrices of s_1 and s_2 that can intersect at $t \in \partial U_2 \cap \mathcal{B}$ are shown in Figs. 6 and

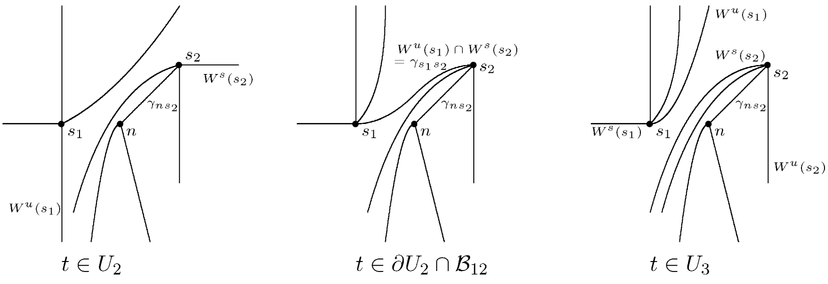

Fig. 7. The bifurcation from U_2 to U_3.

7. In the first case, the saddle-to-saddle separatrix is $\gamma_{s_2 s_1} = W^{\mathrm{u}}(s_2) \cap W^{\mathrm{s}}(s_1)$, so $t \in \mathcal{B}_{21}$, and after the bifurcation $W^{\mathrm{s}}(s_1) = \gamma_{ns_1}$. In the second case, shown in Fig. 7, $\gamma_{s_1 s_2} = W^{\mathrm{u}}(s_1) \cap W^{\mathrm{s}}(s_2)$, so $t \in \mathcal{B}_{12}$. Observe moreover that two separatrices of s_2, among those not intersecting with $W^{\mathrm{u}}(s_1)$ in $\gamma_{s_1 s_2}$, determines in $\mathbb{R}^2$ two disjoint subsets, one containing n and one containing s_1, which implies that after the bifurcation γ_{ns_1} does not appear in the phase portrait of X_t (so the bifurcation is of the type described in Section 4.3). □

Let U_3 be a (maximal) open connected subset such that X_t is structurally stable for all $t \in U_3$ and $\mathcal{B}_{ji} \cap \partial U_2 \cap \partial U_3 \neq \emptyset$.

Proposition 4.16. *For all $t \in U_3$, the phase portrait of X_t does not exhibit the gradient line γ_{ns_j}. At a point $t \in \partial U_3 \cap \mathcal{B}$, two pairs of separatrices, one of s_i and one of s_j, can intersect in a saddle-to-saddle separatrix: in one case, described in Proposition 4.15 and shown in* Fig. 6, *t belongs to $\mathcal{B}_{ji}$, and after the bifurcation the phase portrait of X_t does not exhibit the line γ_{ns_j}; in the other, shown in* Fig. 8, *t belongs to $\mathcal{B}_{ij}$, and after the bifurcation also the phase portrait of X_t does not exhibit the line γ_{ns_j}; both bifurcations are of the type analyzed in Section 4.3 for a family of vector fields with two saddles.*

Proof. The proposition can be proved as done for Proposition 4.15. The saddle-to-saddle separatrices at $t \in \partial U_3 \cap \mathcal{B}$, in the two cases, are shown respectively in Figs. 7 and 8. □

Let U_4 be a (maximal) open connected subset such that X_t is structurally stable for all $t \in U_4$ and $\mathcal{B}_{ji} \cap \partial U_3 \cap \partial U_4 \neq \emptyset$.

Proposition 4.17. *For $t \in U_4$, the phase portrait of X_t does exhibit the gradient line γ_{ns_j} (see* Fig. 8).

Proof. It follows from Proposition 4.16. □

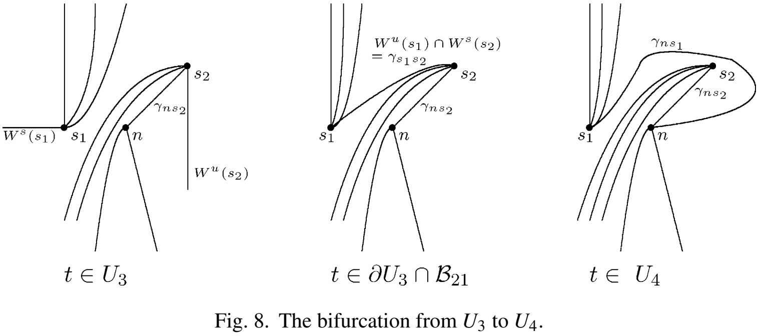

Fig. 8. The bifurcation from U_3 to U_4.

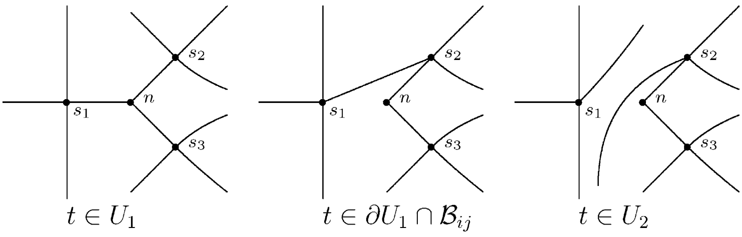

Fig. 9. The bifurcation from U_1 to U_2.

Note that γ_{ns_j} has a different winding around s_i for $t \in U_1$ and $t \in U_4$.

Lemma 4.18. $\mathcal{B}_{1,2} \cap \mathcal{B}_{2,1} = \emptyset$. *The intersection of two components* $\mathcal{B}^1_{ij}$ *and* $\mathcal{B}^2_{ij}$ *of* $\mathcal{B}_{ij}$ *is non-empty provided the saddle-to-saddle separatrices* γ^1_{ij} *and* γ^2_{ij} *of* $\mathcal{B}^1_{ij}$ *and* $\mathcal{B}^2_{ij}$ *are obtained as intersection of the same pair of separatrices of* s_1 *and* s_2.

Proof. See the proof of Lemma 4.13. □

4.5. A family of vector fields with three saddles and one node

Consider a 2-parameter family of gradient vector fields X_x exhibiting three saddles s_1, s_2 and s_3 and an unstable node n. Consider a structurally stable element X_s of the family, such that its phase portrait contains all the gradient lines γ_{ns_i} for $i = 1, 2, 3$, then there exists a neighbourhood U_1 of s such that, for every $t \in U_1$, the phase portrait of X_t is orbitally equivalent to the one of X_s and so it shows the same qualitative features. We can assume U_1 to be maximal among such neighbourhoods of s. At a point $t \in \partial U_1$, the vector field X_t can exhibit one among the saddle-to-saddle separatrices $\gamma_{s_is_j}$, implying $t \in \mathcal{B}_{ij}$. We have $\gamma_{s_is_j} = W^{\mathrm{u}}(s_i) \cap W^{\mathrm{s}}(s_j)$: in this case $W^{\mathrm{s}}(s_j) = \gamma_{ns_j}$, and so at t, where $\gamma_{s_is_j}$ appears, γ_{ns_j} breaks. Moreover, unlike what described for a family of vector fields exhibiting two saddles and a node, the choice of the component of $W^{\mathrm{u}}(s_i)$ is fixed by the presence of γ_{ns_k}, for $k \neq i, j$. In Fig. 9 the case of the saddle-to-saddle separatrix $\gamma_{s_2s_1}$ is outlined.

Let U_2 be a (maximal) open connected subset such that X_t is structurally stable for all $t \in U_2$ and $\mathcal{B}_{ij} \cap \partial U_1 \cap \partial U_2 \neq \emptyset$. For $t \in U_2$, the phase portrait of X_t contains the gradient line γ_{ns_i} and γ_{ns_k} for $k \neq i, j$, but not γ_{ns_j}. Note that two of the separatrices of s_i determine two subsets in the plane, one containing the node n and the saddle s_k, $k \neq i, j$, and the other containing the saddle s_j. This implies that $W^{\mathrm{u}}(s_j) \cap W^{\mathrm{s}}(s_k) = W^{\mathrm{s}}(s_j) \cap W^{\mathrm{u}}(s_k) = \emptyset$, thus $\partial U_2 \cap \mathcal{B}_{jk} = \partial U_2 \cap \mathcal{B}_{kj} = \emptyset$. Instead, ∂U_2 can intersect $\mathcal{B}_{i,j}$, as just described, and also $\mathcal{B}_{j,i}$, $\mathcal{B}_{i,k}$ or $\mathcal{B}_{k,i}$. As to the intersection with $\mathcal{B}_{j,i}$, it holds what already outlined and shown in figures of Section 4.4: indeed, a separatrix of s_j divides the plane into two subsets, one containing the saddle s_k and its separatrices and one containing the pair of separatrices of s_i and s_j intersecting in the saddle-to-saddle separatrix γ_{s_j,s_i}; moreover, when γ_{s_j,s_i} breaks, the phase portrait of X_t does not exhibit γ_{n,s_i}. Instead, for what concerns $\mathcal{B}_{i,k}$, observe that the gradient line γ_{ns_j} does not appear in the phase portrait of X_t, so it follows that $W^{\mathrm{s}}(s_k) = \gamma_{ns_k}$ can intersect both the separatrices defining $W^{\mathrm{u}}(s_i)$, as shown in Figs. 10 and 11. Analogous considerations hold for $\mathcal{B}_{k,i}$.

From Fig. 10 we see that for $t \in U_3$ no intersection is possible between the stable and unstable manifolds of s_1 and s_3, so ∂U_3 can intersect only $\mathcal{B}_{12}$, $\mathcal{B}_{21}$, $\mathcal{B}_{23}$ or $\mathcal{B}_{32}$. From Fig. 11 we see instead that for $t \in U_3$ only $\mathcal{B}_{13}$, $\mathcal{B}_{31}$, $\mathcal{B}_{23}$ and $\mathcal{B}_{32}$ can intersect ∂U_3.

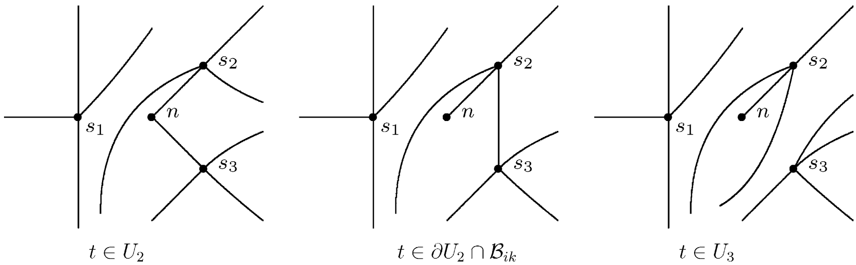

Fig. 10. The bifurcation from U_2 to U_3 (1st case).

We can resume what said in the following, very general, proposition.

Proposition 4.19. *The gradient line $\gamma_{s_i s_j}$ can appear if a component of $W^{\mathrm{u}}(s_i)$ can intersect a component of $W^{\mathrm{s}}(s_j)$. Whether this is possible and which components can actually intersect depends on the separatrices of the third saddle s_k and on the gradient lines $\gamma_{ns_l}, l = 1, 2, 3$, appearing in the phase portrait.*

As to intersection of bifurcation lines, we will analyze some cases when studying the bifurcation locus of perturbations of the elliptic umbilic. The caustic, as we will see, imposes further constraints on the possible intersections.

4.6. A family of vector fields with saddles and nodes

In general, to a family X_x, $x \in \mathbb{R}^2$, of planar vector fields, having nodes $n_1, \ldots, n_\alpha$, and saddles $s_1, \ldots, s_\beta$, we can associate a bifurcation diagram given by the bifurcation locus $\mathcal{B}$ with its components $\mathcal{B}_{ij}$ and $\mathcal{B}_{(ij),(k,l)}$. A proposition similar to 4.19 can be formulated also in this case, explaining where, depending on the phase portrait of X_x, the gradient lines $\gamma_{s_i s_j}$ appear.

Proposition 4.20. *Let $U \subset \mathbb{R}^2$ be an open connected subset such that for every $x \in U$, the vector fields X_x are structurally stable and orbitally equivalent. Then ∂U can intersect $\mathcal{B}_{ij}$ if and only if $W^{\mathrm{u}}(s_i)$ and $W^{\mathrm{s}}(s_j)$ lie in the same connected component determined in the phase portrait of X_x by*:

- *the separatrices of the remaining saddles of X_x*;

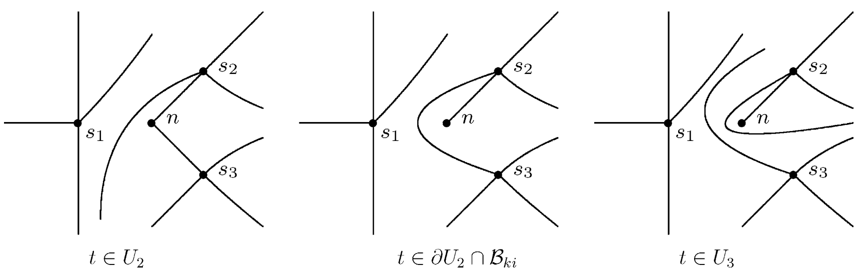

Fig. 11. The bifurcation from U_2 to U_3 (2nd case).

- *the gradient lines $\gamma_{n_k s_j}$ or $\gamma_{s_j n_k}$, respectively if n_k is an unstable or stable node, for $k = 1, \ldots, \alpha$ and $j = 1, \ldots, \beta$.*

As to intersection of bifurcation lines, as said, generically, $\mathcal{B}_{(i,j),(j,k)} = \mathcal{B}_{(i,j)} \cap \mathcal{B}_{(j,k)}$. A necessary condition for $\mathcal{B}_{(i,j),(k,l)}$ to be non-empty is that there exists a vector field whose phase portrait can exhibit both the exceptional gradient lines $\gamma_{s_i s_j}$ and $\gamma_{s_k s_l}$. Moreover, in a neighbourhood $N(t)$ of $t \in \mathcal{B}_{(i,j),(k,l)}$, the saddle-to-saddle separatrices generically break, and the phase portrait of ∇f_x, for $x \in N(t)$, must be recovered continuously from the phase portrait of ∇f_t. In particular, if $t \in \mathcal{B}_{(i,j),(j,k)}$, the exceptional gradient lines $\gamma_{s_i s_j}$ and $\gamma_{s_j s_k}$, can also break in $N(t)$ in such a way to form the saddle-to-saddle separatrix $\gamma_{s_i s_k}$, implying $t \in \bar{\mathcal{B}}_{(i,k)}$. As to intersection of bifurcation lines, we will consider some examples when studying perturbations of the elliptic umbilic.

Definition 4.21. Given a bifurcation diagram, expressions as “the diagram is allowed” or “permitted” will be used to mean that there exists a continuous family of planar vector fields providing the given bifurcation diagram.

For example, a bifurcation diagram, such that $\mathcal{B}_{ij} \cap \mathcal{B}_{ji} \neq \emptyset$, is not allowed.

Observe also that a bifurcation diagram contains information about the existence of non-generic gradient lines but no information about the number and nature of critical points.

4.7. Families of gradient fields

Given an allowed diagram, the problem is now to understand, at least in those example we are concerned with, when there exists a family of vector fields of the form ∇f_x, where $f_x(y) = f(y) - x \cdot y$ with $f : \mathbb{R}^2 \to \mathbb{R}$. The first step is to construct a family of gradient vector fields whose bifurcation diagram is the given one, and then to look for a family with the required dependence from the parameter.

Lemma 4.22. *Suppose that in an open simply connected subset $U \subset \mathbb{R}^2$ the phase portrait of a vector field X does not exhibit any critical points or periodic orbits, then there exists a function f on U such that ∇f is orbitally equivalent to X.*

Proof. Consider the distribution of vector fields $\{X_x\}_{x \in U}$. Since $X(x) \neq 0$ for $x \in U$, we can choose an orthogonal distribution $\{X_x^{\perp}\}_{x \in U}$, and since U is simply connected we can suppose this distribution to be smooth. The hypothesis of Frobenius theorem are satisfied, so through every point $x \in U$ it passes a unique curve integrating the distribution $\{X_x^{\perp}\}_{x \in U}$. In a neighbourhood V of any point $x \in U$ a function f_V can be defined having the curves integrating $\{X_x^{\perp}\}_{x \in U}$ in V as level curves. Since the integral curves of a gradient vector field cross the level sets of its potential orthogonally at points which are not fixed points, it follows that the phase portrait in V of ∇f_V coincides with the phase portrait of $X_{|V}$. Since U is simply connected a function f having the property required can be defined on the whole U. □

Unfortunately, given a phase portrait exhibiting only saddles and nodes, it does not necessarily exist a potential f whose gradient field ∇f exhibits that phase portrait. We can state the following lemma, but not its converse.

Lemma 4.23. *If p is a local maximum, minimum or saddle of f, then p is respectively an unstable node, a stable node or a saddle of ∇f.*

Proof. It is a consequence of the definition of node and saddle of a vector field. □

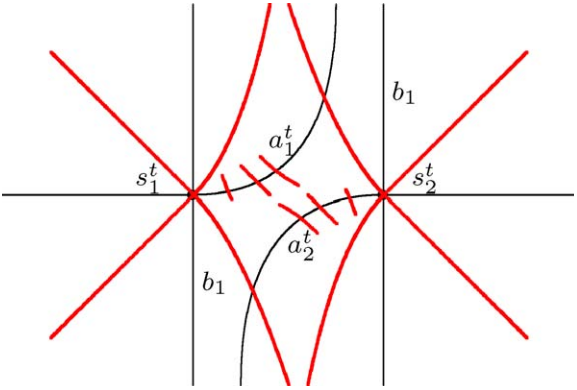

Fig. 12. Phase portrait of $\nabla F(\, , t)$ and level curves of $F(\, , t)$ for $-1 \leq t < 0$.

Lemma 4.24. *Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a function such that ∇f is a Morse–Smale vector field exhibiting two saddles, then there exists a function $F : \mathbb{R}^2 \times [-1, 1] \to \mathbb{R}$ such that $F(\, , -1) = f$, $\nabla F(\, , t)$ is a Morse–Smale vector field exhibiting two saddles for every $t \neq 0$, and $\nabla F(\, , 0)$ has a saddle-to-saddle separatrix.*

Proof. Lemma 4.23 tells which behaviour the level curves of $F(\, , t)$ have in a neighbourhood of saddles, and moreover we know these level curves are orthogonal to the separatrices of the saddles. In Fig. 12, the saddles s_1^t and s_2^t of $\nabla F(\, , t)$, their separatrices, denoted by a_i^t and b_i^t, $i = 1, 2$, and some of the relevant level curves of $F(\, , t)$ (the red lines), for $-1 \leq t < 0$, are shown.

In Fig. 13, the phase portrait of $\nabla F(\, , t)$ and some of the relevant level curves of $F(\, , t)$ for respectively $t = 0$ and $0 < t \leq 1$, are shown. The saddle-to-saddle separatrix is denoted by a^0.

To construct the functions $F(\, , t)$, we first choose two points in $\mathbb{R}^2$, in whose neighbourhoods we define, according to Lemma 4.23, $F(\, , t)$ in such a way that these points are saddles; then, we set $F(x, t) = f(x)$ for every $x \in \mathbb{R}^2 \setminus A$ and $t \in [-1, 1]$, where $A \subset \mathbb{R}^2$ is a neighbourhood, shown in Fig. 14, of the line chosen as the saddle-to-saddle separatrix of $F(0, t)$.

We define the level curves of $F(\, , t)$ in a neighbourhood of a_i^t as the fibres of the normal bundle to a_i^t and extend then $F(\, , t)$ to the whole A (see Fig. 15): this means to require that the derivatives of $F(\, , t)$ in the direction normal to the separatrices a_i^t vanish at any point of the separatrices a_i^t

$$\frac{\partial F(\, , t)}{\partial (a_i^t)^{\perp}}(a_i^t(s)) = 0$$

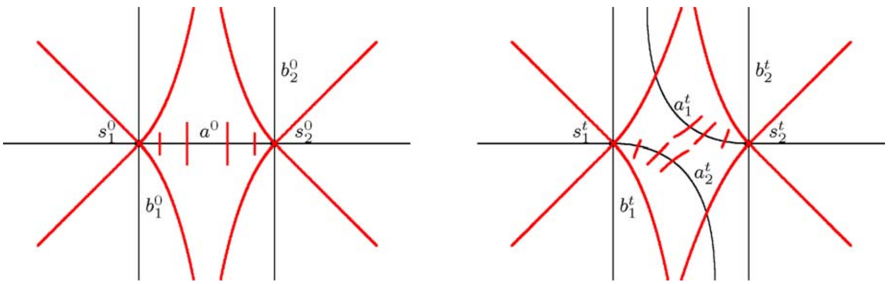

Fig. 13. Phase portrait of $\nabla F(\, , t)$ and level curves of $F(\, , t)$ for $t = 0$ and $0 < t \leq 1$ respectively.

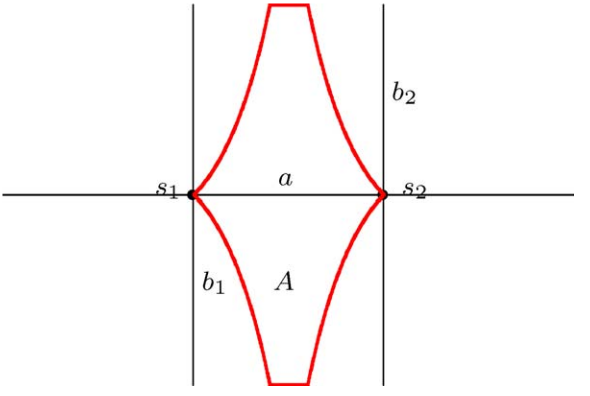

Fig. 14. The subset A.

By construction, $\nabla F(\, , t)$ has the required properties. □

Observe that, for each t, the conditions defining the function $F(\, , t)$ concerns only those points which we choose as critical points or belonging to separatrices in A.

Corollary 4.25. *Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a function such that ∇f is a Morse–Smale vector field exhibiting two saddles, then there exists a function $F : \mathbb{R}^2 \times D^2 \to \mathbb{R}$, where $D^2 = \{t_1^2 + t_2^2 \leq 1\} \subset \mathbb{R}^2$, such that $F(\, , 0, -1) = f$, $\nabla F(\, , t_1, t_2)$ is a Morse–Smale vector field exhibiting two saddles for every $t_2 \neq 0$, and $\nabla F(\, , t_1, 0)$ has a saddle-to-saddle separatrix.*

Proof. The requirement that the family $F(\, , t_1, t_2)$ exhibits a saddle-to-saddle separatrix along the subset $\{t_2 = 0\} \subset D^2$ is compatible with what said about the dimension of components of the bifurcation locus. The proof is as for Lemma 4.24, where it is not used the fact that the parameters space has dimension 1. □

Lemma 4.26. *Given a bifurcation diagram exhibiting a bifurcation line $\mathcal{B}$, there exists a family of gradient vector fields in a neighbourhood of $\mathcal{B}$ having only two saddles and with $\mathcal{B}$ as associated bifurcation diagram.*

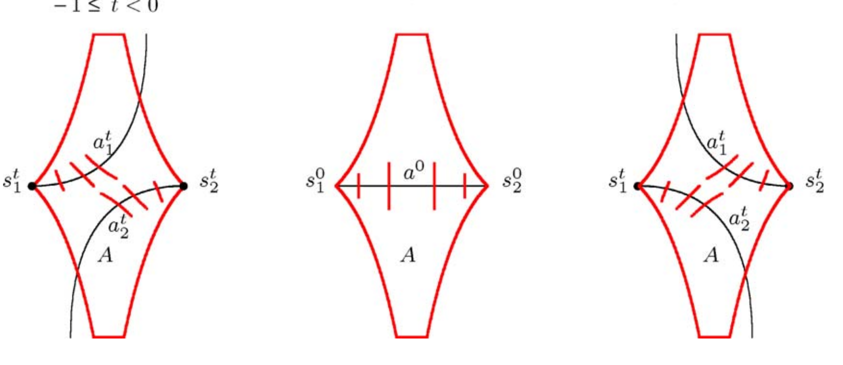

Fig. 15. The construction of F.

Proof. By Lemma 4.23 we choose a family of functions having two saddles points and we apply Corollary 4.25. □

The following corollary makes global the result of Lemma 4.26.

Corollary 4.27. *Given an allowed bifurcation diagram $\mathcal{B}$, there exists a family of gradient vector fields having $\mathcal{B}$ as associated bifurcation diagram.*

Proof. It is enough to apply Lemma 4.26 in a neighbourhood of each component $\mathcal{B}_{ij}$ of the bifurcation locus. □

The second step is to prove that the family of vector fields of Corollary 4.27 can be chosen depending linearly on the parameter x.

Definition 4.28. A bifurcation diagram $\mathcal{M}$ is a subset of a bifurcation diagram $\mathcal{N}$ if the bifurcation locus of $\mathcal{M}$ is a subset of the bifurcation locus of $\mathcal{N}$.

Definition 4.29. Two bifurcation diagrams $\mathcal{M}$ and $\mathcal{N}$ are equivalent if there exists a diffeomorphism of $\mathbb{R}^2$ mapping caustic and bifurcation locus of $\mathcal{M}$ onto those of $\mathcal{N}$.

Theorem 4.30. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the generating function of a Lagrangian submanifold L, suppose $0 \in \mathbb{R}^2$ is a critical point of f and W is a compact neighbourhood of 0. Given a bifurcation diagram $\mathcal{M}$ containing a caustic K and a bifurcation locus $\mathcal{B}$, such that the number of connected components of $f(W) \setminus (\mathcal{B} \cup K)$ is finite, $\mathcal{M}$ is allowed and K is diffeomorphic to the caustic of a small perturbation of f in W, then, if W is sufficiently small, there exists a generating function $\tilde{f} = f + f'$, such that f' is supported on W, and whose associated bifurcation diagram, restricted to W, contains a subdiagram equivalent to $\mathcal{M}$ restricted to W.*

Proof. Let U_i be the connected components of $f(W) \setminus (\mathcal{B} \cup K)$: note that U_i is open and choose a point $x_i \in U_i$. The bifurcation diagram $\mathcal{M}$, being allowed, prescribes the classes of orbital equivalence of the phase portrait of $\nabla \tilde{f}_{x_i}$ in each subset U_i, so we define a function $\tilde{f}_{x_i}$ such that number and nature of its critical points and behaviour of gradient lines joining each pair of these critical points are as assigned by $\mathcal{M}$. The function $\tilde{f}_{x_i}$ is constructed as $F(\, , t)$ in the proof of Proposition 4.24, so, for $\tilde{f}_{x_i}$ to satisfy the required conditions, it is enough to define it in a neighbourhood V_i of the chosen critical points and relevant gradient lines. Observe that we can assume $V_i \cap V_j = \emptyset$ for $i \neq j$, since the number of V_i's is finite. Define now $\tilde{f}$ on $\cup U_i$ as $\tilde{f}(y) = \tilde{f}_{x_i}(y) + x_i y$ and extend it to the whole W. Note that for every $\epsilon > 0$, since $\nabla f(0) = 0$ and the conditions $\tilde{f}$ has to satisfy concern its gradient $\nabla \tilde{f}$, if W is sufficiently small, then $|f'| < \epsilon$, where $| \, |$ is a quasi-norm associated with the Whitney topology of $C^\infty(\mathbb{R}^2)$. Observe also that, since $\nabla \tilde{f}_{x_i}$ is structurally stable on V_i, then there exists a neighbourhood U_i' of x_i in U_i such that $\nabla \tilde{f}_x$ is orbitally equivalent to $\nabla \tilde{f}_{x_i}$ for all $x \in U_i'$. The function $\tilde{f}$ has the required properties: indeed, by choosing ϵ sufficiently small, the caustic of $\tilde{f}$ is diffeomorphic to the caustic K in $\mathcal{M}$; moreover, take a path $c : [-1, 1] \to W$ such that $c(0) = x_i$, $c(1) = x_j$ and $c(t) \in \mathcal{B}$ for some $t \in [-1, 1]$, then, as in the proof of Proposition 4.10, there exists a point $t' \in [-1, 1]$ such that $c(t')$ is a bifurcation point for $\tilde{f}$. □

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